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Multivector Dirac Equation and $\mathbb{Z}_2$-Gradings of Clifford Algebras

R. A. Mosna,[1] D. Miralles,[2,3] and J. Vaz Jr.[4,5]

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We generalize certain aspects of Hestenes's approach to Dirac theory to obtain multivector Dirac equations associated to a large class of representations of the gamma matrices. This is done by replacing the usual even/odd decomposition of the space-time algebra with more general $\mathbb{Z}_2$-gradings. Some examples are given and the chiral case, which is not addressed by the usual approach, is considered in detail. A Lagrangian formulation is briefly discussed. A relationship between this work and certain quaternionic models of the (usual) quantum mechanics is obtained. Finally, we discuss under what conditions the Hestenes's form can be recovered and we suggest a geometrical interpretation for the corresponding situation.

KEY WORDS: Dirac theory; Dirac–Hestenes equation; Clifford algebras; $\mathbb{Z}_2$-grading; spinors.

1. INTRODUCTION

As soon as Dirac proposed the equation bearing his name, alternative multivector formulations of it have been proposed with either physical or mathematical motivations. As examples, one can mention the works of Ivanenko Fock, Landav, Proca, Eddington, Schönberg, Kähler, and Hestenes. In general, these approaches seek a better understanding of the geometric foundations underlying both the Dirac theory and the concept of spinor fields, besides new applications. For example, in the context of the Dirac–Kähler equation, spinor fields are represented by differential forms. Then, the well-known duality between these objects and elements from algebraic topology leads to a straightforward lattice approximation for fermion fields (Becher and Joos, 1982; Rabin, 1982).

[1] Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas, Campinas, SP, Brazil.
[2] Departament de Física Fonamental, Universitat de Barcelona, Barcelona, Catalonia, Spain.
[3] Laboratori de Física Matemática, Societat Catalana de Física, IEC, Catalunya, Spain.
[4] Departamento de Matemática Aplicada, Universidade Estadual de Campinas, Campinas, SP, Brazil.
[5] To whom correspondence should be addressed at Departamento de Matemática Aplicada, Universidade Estadual de Campinas, CP 6065, 13081-970, Campinas, SP, Brazil; e-mail: vaz@ime.unicamp.br.

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On the other hand, in the Hestenes approach, extensive use of the space-time algebra (the real Clifford algebra of space-time $\mathcal{C}l_{1,3}(\mathbb{R})$) is made to give spinors a geometrical interpretation. In this context, the spinor Ψ is an element of the usual even part $\mathcal{C}l^{+}_{1,3}(\mathbb{R})$ of that algebra (see later), and this leads to an elegant canonical decomposition for Ψ, which generalizes the polar decomposition of complex numbers. Also, the spinor acquires operatorial attributes: its action on the basis multivectors $e_{\mu_1\cdots\mu_k}$ of the observer's reference frame $\{e_\mu\}$ yields the observables of the theory. For instance, the current density $j = j^\mu e_\mu$ and the magnetization density $M = \frac{1}{2}M^{\mu\nu}e_{\mu\nu}$ are given by $j = \Psi e_0\tilde{\Psi}$ and $M = \frac{e}{2m}\Psi e_{21}\tilde{\Psi}$ (with $\hbar = c = 1$), where $(\cdot)^{\tilde{}}$ is the reversion operation (see later). Moreover, Ψ belongs to the same algebra as the observables of the theory, i.e., the Clifford algebra unifies, in a certain sense, the concepts of state and operator. The corresponding multivector Dirac equation obtained in this context is the so-called Dirac–Hestenes equation (Hestenes, 1967, 1995): $\partial\Psi e_{21} = m\Psi e_0$, where $\partial = e^\mu\partial_\mu$. Besides providing an elegant formulation of Dirac theory, such an approach leads to some computational advantages (Gull *et al.*, 1996). Nevertheless, it is our opinion that it has not been fully explored, as we intend to show in the following.

As we review in the next section, the Hestenes's formulation can be obtained from the usual one by using a particular isomorphism $\rho_{\rm st} : \mathcal{C}l_{1,3}(\mathbb{C}) \to \mathcal{M}(4, \mathbb{C})$, where $\mathcal{M}(4, \mathbb{C})$ is the algebra of 4×4 complex matrices. Given a reference frame $\{e_\mu\}$ (which corresponds to an observer) in Minkowski space $\mathbb{M}$, such isomorphism is given by $\rho_{\rm st}(e_\mu) = \gamma^{\rm st}_\mu$, where $\{\gamma^{\rm st}_\mu\}$ are the gamma matrices in the standard representation. As it is well known, the Dirac–Hestenes equation is independent of $\{e_\mu\}$, for another choice $\{e'_\mu\}$ must be related to the old one by $e'_\mu = Ue_\mu\tilde{U}$, with $U \in \mathit{Spin}_+(1, 3)$, giving $\partial\Psi' e'_{21} = m\Psi' e'_0$, where $\Psi' = \Psi\tilde{U}$ (Hestenes, 1995). On the other hand, in the usual matrix formulation of the Dirac theory, we have a complete freedom of choice for the representation in which the gamma matrices are. As the underlying algebra is simple, it turns out that all such choices are given by an internal transformation of the form $\mathsf{S}\gamma^{\rm st}_\mu\mathsf{S}^{-1}$, where S is an arbitrary invertible matrix. Restricting ourselves to transformations where S is unitary, we have (Messiah, 1961)

$$\begin{aligned} \gamma^{\rm st}_\mu &\mapsto \gamma_\mu = \mathsf{S}\gamma^{\rm st}_\mu\mathsf{S}^{-1}, \\ |\psi\rangle &\mapsto \mathsf{S}|\psi\rangle. \end{aligned} \tag{1}$$

By varying S we have the standard, chiral, Majorana, or any other representation. Let us fix a reference frame $\{e_\mu\}$ corresponding to a given observer. Also, let us associate to each such $\{\gamma_\mu\}$ a new isomorphism $\rho : \mathcal{C}l_{1,3}(\mathbb{C}) \to \mathcal{M}(4, \mathbb{C})$, defined by

$$\rho(e_\mu) := \gamma_\mu.$$

We can easily transfer the arbitrariness in $\{\gamma_\mu\}$ to an arbitrariness in the choice of the ONS $\{e_\mu\}$ (see sec. 1.1): by defining $S := \rho_{\text{st}}^{-1}(\mathsf{S}) \in \mathcal{C}l_{1,3}(\mathbb{C})$, we have $\rho(e_\mu) = \gamma_\mu = \mathsf{S}\gamma_\mu^{\text{st}}\mathsf{S}^{-1} = \mathsf{S}\rho_{\text{st}}(e_\mu)\mathsf{S}^{-1} = \rho_{\text{st}}(Se_\mu S^{-1}) = \rho_{\text{st}}(e'_\mu)$. Thus, to see the effect of an arbitrary $\{\gamma_\mu\}$, we can alternatively fix the representation ρ and vary the ONS $\{e_\mu\}$ by $e_\mu \to Se_\mu S^{-1}$, with $S \in \mathcal{C}l_{1,3}(\mathbb{C})$. The unitarity of S has the following counterpart in S. Let us induce a (ρ-dependent) Hermitian conjugation in $\mathcal{C}l_{1,3}(\mathbb{C})$ by $S^\dagger = \rho^{-1}(\mathsf{S}^\dagger) = \rho^{-1}(\rho(S)^\dagger)$. Then $\mathsf{S}^{-1} = \mathsf{S}^\dagger \Leftrightarrow S^{-1} = S^\dagger$.

Note that the Hestenes's formulation is *not invariant* under the above transformation for e_μ, i.e. under[6]

$$
\begin{aligned}
e_\mu &\mapsto e'_\mu = Se_\mu S^{-1}, \\
\Psi &\mapsto \Psi' = \Psi S^{-1}.
\end{aligned}
\tag{2}
$$

Indeed, for arbitrary S, Ψ' no longer belongs to the even part of the algebra and, for $S \notin Spin_+(1,3)$, e'_μ is not even a 1-vector. Hence, we see that this "difficulty" arises because we are restricted to a fixed $\mathbb{Z}$-grading of the underlying vector space structure of $\mathcal{C}l_{1,3}(\mathbb{R}) = \oplus_k \Lambda_k(\mathbb{R}^{1,3})$, the one in which the tangent vectors of space-time are elements of $\Lambda_1(\mathbb{R}^{1,3}) \subseteq \mathcal{C}l_{1,3}(\mathbb{R})$. This is certainly the most natural choice but not the unique one, as it has already been remarked by Fauser (2001), "One knows that different $\mathbb{Z}_n$-gradings can produce quite different spinor modules. This fact renders the unquestioned multivector structure as a peculiar one. A careful study of the representation theory and their dependence on gradings in such cases is required."

In the following, we circumvent this difficulty by allowing more general $\mathbb{Z}_2$-gradings $\mathcal{C}l_0 \oplus \mathcal{C}l_1$ to the space-time algebra, i.e., by generalizing its usual decomposition in terms of even/odd parts. Then, the spinor space is given by the generalized even part $\mathcal{C}l_0$, of $\mathcal{C}l_{1,3}(\mathbb{R})$, which can be isomorphic to either $\mathcal{M}(2,\mathbb{C})$ or $\mathbb{H} \oplus \mathbb{H}$ (as algebras). As a result, we obtain multivector Dirac equations corresponding to a large class of representations of the gamma matrices. We work out the standard, Majorana, and chiral cases, giving special attention to the last one. In this context, a neat characterization for chirality in terms of the even/odd parts of the new spinor space is given. A Lagrangian formulation is also briefly discussed.

[6] Although we have *transferred* an arbitrariness in $\mathcal{M}(4,\mathbb{C})$ to an arbitrariness in $\mathcal{C}l_{1,3}(\mathbb{C})$, we stress that the transformations (1) and (2) are not the counterparts of each other. In fact, every operation acting on the left side of $|\psi\rangle$ in the matrix case has the well-defined analogue in the operatorial approach:

$$
\begin{aligned}
|\psi\rangle &\to \gamma_\mu^{\text{st}}|\psi\rangle \mapsto \Psi \to e_\mu \Psi e_0, \\
|\psi\rangle &\to i|\psi\rangle \mapsto \Psi \to \Psi e_{21}, \quad i = \sqrt{-1}.
\end{aligned}
$$

On the other hand, the transformation $\Psi \to \Psi S^{-1}$, with S acting from the right, has no analogue in the matrix case. This kind of transformations has already been considered by a number of authors (Chisholm and Farwell, 1999; Hestenes, 1982), although in different contexts.

As a by-product, we show that Rotelli's quaternionic formulation of the (usual) quantum mechanics (Rotelli, 1989) can be derived from our approach. This is done by giving a natural derivation for the complex projection of his scalar product.[7] Also, we show that the multivector Dirac equations discussed here provide a natural way to obtain gamma matrix representations in terms of the enhanced $\mathbb{H}$-general linear group $GL(2, \mathbb{H}) \cdot \mathbb{H}^*$ (Harvey, 1990), which comprises matrix multiplication from the left and scalar multiplication from the right, as in de Leo (2001).

We stress that the alternative $\mathbb{Z}_2$-gradings discussed earlier are defined without disturbing the aforementioned multivector structure $\oplus_k \Lambda_k(\mathbb{R}^{1,3})$ of $\mathcal{C}l_{1,3}(\mathbb{R})$. In other words, one can still interpret elements of Λ_0 as scalars, elements of Λ_1 as tangent vectors of space-time, and so on. On the other hand, the usual Hestenes's form of the Dirac equation can be recovered if we allow arbitrary $\mathbb{Z}$-gradings for the underlying vector space structure of $\mathcal{C}l_{1,3}(\mathbb{R})$ (i.e. arbitrary multivector structures), as we show in Section 3. Such alternative gradings have already been considered in the literature, although in a different context (see Fauser and Ablamowicz (2000) for an excellent discussion on this issue). In our case, we argue that this situation can be interpreted as if different representations $\{\gamma_\mu\}$ determine different slices of the space-time algebra $\mathcal{C}l_{1,3}(\mathbb{R})$, each of them corresponding to a copy of the Minkowski space $\mathbb{M}$. This points to a connection between the present work and Pezzaglia's polydimensional physics program (Pezzaglia, 1999). Finally, in the Appendix, we elaborate on the spinor maps used in the main text.

In the process of writing this paper, we became aware that the construction of the spinor space $\mathcal{C}l_0$ employed by us is reminiscent of a general method of representation of Clifford algebras introduced by Dimakis (1989). In that paper, the author introduces representations of arbitrary Clifford algebras on subspaces obtained by successive $\mathbb{Z}_2$-gradings of the original algebra. As a result, these subspaces are always real Clifford algebras by themselves. For $\mathcal{C}l_{1,3}(\mathbb{R})$, the spinor spaces obtained by Dimakis are the same as ours, but *this is not the general rule.* In fact, our approach leads to more general spinor spaces, which can be classified (Mosna *et al.*, in preparation) by

$$\mathcal{C}l_0 \cong \mathcal{C}l_{p_0,q_0} \otimes \mathcal{C}l^+_{p-p_0,q-q_0},$$

where $p_0(q_0)$ is the number of independent even[8] 1-vectors squaring to $+1$ (-1) (note that for $p_0 = q_0 = 0$, this expression reduces to $\mathcal{C}l_0 = \mathcal{C}l^+_{p,q}(\mathbb{R})$, as expected). As an example, for the Clifford algebra $\mathcal{C}l_{3,0}(\mathbb{R})$, which is related to Pauli theory

[7] It is important to note that such a quaternionic formulation (Rotelli, 1989) (see also de Leo, 2001) bears no relation to the much more general program due to Finkelstein *et al.* (1962), Emch (1963), and Adler (1995), where a truly quaternionic valued scalar product is used instead of a complex valued one. It is not a surprise then that the quaternionic formulation we refer to in the main text can be actually derived instead of being postulated.

[8] Of course, in this context even means belonging to $\mathcal{C}l_0$.

in the same way as $\mathcal{C}l_{1,3}(\mathbb{R})$ is related to Dirac theory, our method leads to spinor spaces isomorphic to $\mathbb{H}$, $\mathcal{M}(2, \mathbb{R})$, and $\mathbb{C} \oplus \mathbb{C}$. As $\mathbb{C} \oplus \mathbb{C}$ is not a real Clifford algebra, this case is not given by Dimakis's method. Anyway, the subject of the present paper is different from Dimakis (1989), as we are mainly interested in the general multivector Dirac equations outlined earlier, as well as in their consequences. A more abstract discussion on alternative $\mathbb{Z}_2$-gradings of Clifford algebras (as well as some applications) will be considered in a separate paper (Mosna *et al.*, in preperation).

1.1. Algebraic Preliminaries and Notation

We start by establishing some notation. We say that a vector space V is graded by an Abelian group G if V is expressible as a direct sum $V = \oplus_i V_i$ of subspaces labelled by elements $i \in G$ (Benn and Tucker, 1987). Here we consider only the cases when G is given by $\mathbb{Z}$ or $\mathbb{Z}_2$. In this case, the elements of V_i are called homogeneous of degree i and we define $\deg(v) = i$ if $v \in V_i$. We say that an algebra $\mathcal{A}$ is graded by G if (a) the subjacent vector space of $\mathcal{A}$ is a G-graded vector space and (b) the algebra product satisfies $\deg(ab) = \deg(a) + \deg(b)$.

As usual, let $\mathbb{R}^{p,q}$ be the model of an n-dimensional real vector space endowed with a nondegenerate symmetric metric g of signature (p, q), where $n = p + q$. The Grassmann algebra over $\mathbb{R}^{p,q}$ will be denoted by $\Lambda(\mathbb{R}^{p,q}) = \oplus_{k=0}^{n} \Lambda_k(\mathbb{R}^{p,q})$. We note that $\Lambda(\mathbb{R}^{p,q})$ is an example of a $\mathbb{Z}$-graded algebra (under the exterior product $\wedge$). We denote the projection of a multivector $a = a_0 + a_1 + \cdots + a_n$, with $a_k \in \Lambda_k(\mathbb{R}^{p,q})$, on its p-vector part by $\langle a \rangle_p := a_p$. The parity operator $(\cdot)\hat{}$ is defined as the algebra automorphism generated by the expression $\hat{v} = -v$ on vectors $v \in \mathbb{R}^{p,q}$. The reversion $(\cdot)^{\sim}$ is the algebra anti-automorphism generated by the expression $\tilde{v} = v$ on vectors $v \in \mathbb{R}^{p,q}$. It follows that $\hat{a} = (-1)^k a$ and $\tilde{a} = (-1)^{[k/2]} a$ if $a \in \Lambda_k(\mathbb{R}^{p,q})$, where $[m]$ denotes the integer part of m. Given $a = u_1 \wedge \cdots \wedge u_k$ and $b = v_1 \wedge \cdots \wedge v_1$ with $u_i, v_j \in \mathbb{R}^{p,q}$, the expressions $g(a, b) = \det(g(u_i, v_j))$, if $k = l$, and $g(a, b) = 0$, if $k \neq l$, extend g to $\Lambda(\mathbb{R}^{p,q})$. Also, the left contraction $\lrcorner$ on the Grassmann algebra is defined by $g(a \lrcorner b, c) = g(b, \tilde{a} \wedge c)$ for $a, b, c \in \wedge(\mathbb{R}^{p,q})$.

The Clifford product between a vector $v \in \mathbb{R}^{p,q}$ and a multivector a in $\Lambda(\mathbb{R}^{p,q})$ is given by $va = v \wedge a + v \lrcorner a$. This is extended by linearity and associativity to all of $\Lambda(\mathbb{R}^{p,q})$. The resulting algebra is the so-called Clifford algebra $\mathcal{C}l_{p,q}(\mathbb{R})$. Note that $\mathcal{C}l_{p,q}(\mathbb{R})$ is a $\mathbb{Z}$-graded vector space (for it is linear isomorphic to $\Lambda(\mathbb{R}^{p,q})$), but it is not a $\mathbb{Z}$-graded algebra as, for example, the Clifford product between two 1-vectors is a sum of elements of degrees 0 and 2. Nevertheless, there are (infinite) $\mathbb{Z}_2$-gradings which are compatible with the Clifford product structure. For instance, the usual $\mathbb{Z}_2$-grading of $\mathcal{C}l_{p,q}(\mathbb{R})$ is given by $\mathcal{C}l^+_{p,q}(\mathbb{R}) \oplus \mathcal{C}l^-_{p,q}(\mathbb{R})$ where $\mathcal{C}l^+_{p,q}(\mathbb{R}) = \oplus_{k \text{ even}} \Lambda_k(\mathbb{R}^{p,q})$ and $\mathcal{C}l^-_{p,q}(\mathbb{R}) = \oplus_{k \text{ odd}} \Lambda_k(\mathbb{R}^{p,q})$. We denote the complexification of $\mathcal{C}l_{p,q}(\mathbb{R})$ by $\mathcal{C}l_{p,q}(\mathbb{C}) = \mathcal{C}l_{p,q}(\mathbb{R}) \otimes \mathbb{C}$ (of course, all the $\mathcal{C}l_{p,q}(\mathbb{C})$ with fixed

$p + q$ are isomorphic as complex algebras). The parity operator and the reversion extend naturally to the complexified case. The complex conjugation in $\mathcal{C}l_{p,q}(\mathbb{C})$ will be denoted by $(\cdot)^*$.

In terms of the Clifford product, we have $g(a, b) = \langle \tilde{a} b \rangle_0$, for a, b arbitrary multivectors in $\mathcal{C}l_{p,q}(\mathbb{R})$. For *1-vectors*, this reduces to $2g(x, y) = xy + yx$, $x, y \in \mathbb{R}^{p,q}$, but this expression does not hold for arbitrary multivectors (except for particular cases). Given an orthonormal basis $\{e_i\}$ of $\mathbb{R}^{p,q}$, we have of course $e_i e_j + e_j e_i = 2g_{ij}$, where $g_{ij} = g(e_i, e_j)$. When a subset $\{E_i\}$ of $\mathcal{C}l_{p,q}(\mathbb{R})$, *not necessarily composed of vectors* in $\mathbb{R}^{p,q}$, satisfy the analogous property $E_i E_j + E_j E_i = 2g_{ij}$, we say that $\{E_i\}$ is an ONS (for orthonormal set). As we discussed earlier, in this case the expression $E_i E_j + E_j E_i$ does not necessarily represent the scalar product $g(E_i, E_j)$ between E_i and E_j (see also Section 3). If c is an invertible element of $\mathcal{C}l_{p,q}(\mathbb{R})$, the definition $E_i = c e_i c^{-1}$ gives an example of an ONS.

2. MULTIVECTOR DIRAC EQUATIONS

Let us review one particular way in which Hestenes's approach to Dirac theory emerges from the matricial one. We will only consider a free particle (the introduction of a vector potential can be done by introducing the usual covariant derivative $\partial_\mu \mapsto \partial_\mu + ieA_\mu$). In this case, the Dirac equation in its traditional form (Bjorken and Drell, 1964) is given by

$$i\gamma^\mu \partial_\mu |\psi\rangle = m|\psi\rangle. \tag{3}$$

Here, $|\psi\rangle = (\psi_1\ \psi_2\ \psi_3\ \psi_4)^t$ denotes a column vector in $\mathbb{C}^4$ and $\{\gamma_\mu\}$ is a set of gamma matrices in some representation. These column spinors can be included in $\mathcal{M}(4, \mathbb{C})$ (4×4 complex matrices) by considering, for instance

$$|\psi\rangle = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} = |\psi\rangle\, \mathsf{P}, \quad \text{where } \mathsf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

(of course, we could have chosen any column for such inclusion). Therefore, $|\psi\rangle$ can be assumed to live in the minimal left ideal $\mathcal{M}(4, \mathbb{C})\mathsf{P}$.

As it is well known, $\mathcal{M}(4, \mathbb{C})$ is (noncanonically) isomorphic to $\mathcal{C}l_{1,3}(\mathbb{C})$, the complexified Clifford algebra of the space-time. Let $\rho : \mathcal{C}l_{1,3}(\mathbb{C}) \to \mathcal{M}(4, \mathbb{C})$ be such an arbitrary but fixed isomorphism. Then $P := \rho^{-1}(\mathsf{P})$ is a primitive idempotent and the corresponding minimal left ideal $\mathcal{C}l_{1,3}(\mathbb{C})P$ can be considered as the space of spinors. Elements belonging to this ideal are called *algebraic spinors*. Note that we *started* from a representation ρ of $\mathcal{C}l_{1,3}(\mathbb{C})$ and then we *calculated* $P = \rho^{-1}(\mathsf{P})$, which in turn determined the space of algebraic spinors $\mathcal{C}l_{1,3}(\mathbb{C})P$. In the next subsection we specialize to the Hestenes's choice for ρ and after that we generalize it.

In the following, let $\{e_\mu\}$ be an arbitrary but fixed reference frame corresponding to a given observer. In other words, we are considering the physics from the viewpoint of an arbitrary but fixed observer.

2.1. The Dirac–Hestenes Equation

The Hestenes's choice for ρ, which we denote by $\rho_{\rm st}$, is given by[9]

$$\rho_{\rm st}(e_\mu) = \gamma_\mu^{\rm st}, \tag{5}$$

where $\gamma_\mu^{\rm st}$ are the Dirac matrices in the *standard representation* (Itzykson and Zuber, 1980):

$$\rho_{\rm st}(e_0) = \gamma_0^{\rm st} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \rho_{\rm st}(e_k) = \gamma_k^{\rm st} = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \tag{6}$$

(σ_k are the Pauli matrices). It follows that $P = P_{\rm st} = \frac{1}{2}(1 + e_0)\frac{1}{2}(1 + i e_{12})$. By using the fact that $iP_{\rm st} = P_{\rm st}e_{21}$, Hestenes then eliminates the need for complex numbers in the Dirac equation, which can now be written in a minimal left ideal in $\mathcal{C}l_{1,3}(\mathbb{R})$ (*real* algebra):

$$\gamma^\mu \partial_\mu \Phi e_{21} = m\Phi, \quad \Phi \in \mathcal{C}l_{1,3}(\mathbb{R})\frac{1}{2}(1 + e_0).$$

Separation of this equation in its even and odd parts with respect to the main involution $\hat{(\cdot)}$ leads directly to the Dirac–Hestenes equation:

$$\partial \Psi e_{21} = m\Psi e_0, \quad \Psi \in \mathcal{C}l_{1,3}^+(\mathbb{R}). \tag{7}$$

where $\partial = e^\mu \partial\mu$. As we have already said in the Introduction, the spinor Ψ has operatorial attributes, acting on the multivectors $e_{\mu_1 \cdots \mu_k}$ to yield the observables of the theory. For that reason, we say that Ψ is an *operator spinor*. Also, there is a canonical decomposition for Ψ in the form

$$\Psi = \sqrt{\varrho} e^{\frac{\beta}{2} e_{0123}} R,$$

where ϱ and β are real numbers and R is an element of $Spin_+(1, 3)$, the double cover of the restricted Lorentz transformations. For example, Ψ acts on e_0 as a Lorentz transformation together with a probability density (Hestenes, 1995) to yield the physical current $j = \Psi e_0 \tilde{\Psi} = \varrho R_{e_0} \tilde{R}$. The parameter β only gives a nontrivial effect when Ψ acts on an even multivector.

[9] A more precise notation for ρ would have to mention the frame $\{e_\mu\}$, as ρ is observer dependent (see remark at the end of the previous section).

2.2. Generalizing Hestenes's Approach

The Hestenes's approach makes direct use of the standard representation of the gamma matrices. Let us now derive operatorial versions of the Dirac equation corresponding to a larger class of representations. We detail the calculations for the sake of completeness and clarity, but we stress that we are just following the steps of Lounesto (1996), with the introduction of appropriate modifications coming from this more general setting.

An arbitrary representation $\{\gamma_\mu\}$ for the gamma matrices is related to the standard one by an internal transformation of the form $\gamma_\mu = \mathsf{S}\gamma_\mu^{\text{st}}\mathsf{S}^{-1}$. With unitary matrices, we have the corresponding transformation $|\psi\rangle \mapsto \mathsf{S}|\psi\rangle$ for the column spinors. Let us associate to each such $\{\gamma_\mu\}$ a new isomorphism $\rho : \mathcal{C}l_{1,3}(\mathbb{C}) \mapsto \mathcal{M}(4, \mathbb{C})$, defined by (cf (5))

$$\rho(e_\mu) := \gamma_\mu.$$

As we advanced in the Introduction, we can transfer the arbitrariness in the choice of $\{\gamma_\mu\}$ to an arbitrariness in the choice of the ONS $\{e_\mu\}$: defining $S := \rho_{\text{st}}^{-1}(\mathsf{S}) \in \mathcal{C}l_{1,3}(\mathbb{C})$, we have $\rho(e_\mu) = \gamma_\mu = \rho_{\text{st}}(e'_\mu)$, where $e'_\mu = Se_\mu S^{-1}$. Thus, the effect of an arbitrary $\{\gamma_\mu\}$ is emulated by varying the ONS $\{e_\mu\}$ by $e_\mu \to Se_\mu S^{-1}$, with $S \in \mathcal{C}l_{1,3}(\mathbb{C})$. By defining $ad_S : \mathcal{C}l_{1,3}(\mathbb{C}) \to \mathcal{C}l_{1,3}(\mathbb{C})$, $ad_S(a) = SaS^{-1}$, we have then

$$\rho = \rho_{\text{st}} \circ ad_S. \tag{8}$$

Let us now apply ρ^{-1} to (3) with $|\psi\rangle$ in the form (4). We then have

$$ie_\mu \partial^\mu \psi = m\psi, \tag{9}$$

where $\psi = \rho^{-1}(|\psi\rangle)$. Remember from (4) that $|\psi\rangle = |\psi\rangle\mathsf{P}$ and thus $\psi = \psi P$, with $P = \rho^{-1}(\mathsf{P})$. Hence, ψ is an algebraic spinor belonging to the minimal left ideal determined by this new P. It follows from (8) that $P = S^{-1}P_{\text{st}}S = \frac{1}{2}(1 + u)\frac{1}{2}(1 + i\sigma)$, where $u = S^{-1}e_0 S$ and $\sigma = S^{-1}e_{12}S$ are commuting elements such that $u^2 = 1$ and $\sigma^2 = -1$. In general, u and σ don't have to be real but, to the best of our knowledge, every set of gamma matrices considered in physics satisfies this condition.[10]

So, for the sake of simplicity, we assume that the idempotent $P = \rho^{-1}(\mathsf{P})$ can always be written as

$$P = \frac{1}{2}(1 + u)\frac{1}{2}(1 + i\sigma), \tag{10}$$

where u and σ are real commuting elements such that $u^2 = 1$, $\sigma^2 = -1$.

[10] Of course, this does not mean that the matrix S in (1) is real. Indeed, $\rho_{\text{st}} : \mathcal{C}l_{1,3}(\mathbb{C}) = \mathbb{C} \otimes \mathcal{C}l_{1,3}(\mathbb{R}) \to \mathcal{M}(4, \mathbb{C})$ is such that $\rho_{\text{st}}(i \otimes a) = i\rho_{\text{st}}(1 \otimes a)$ for $a \in \mathcal{C}l_{1,3}(\mathbb{R})$, but $\rho_{\text{st}}(1 \otimes a)$ is not necessarily a real matrix (as it is the case of e_2 and γ_2^{st}, for instance).

Let us define four mutually annihilating idempotents (note that $P_1 = P$):

$$P_1 = \frac{1}{2}(1+u)\frac{1}{2}(1+i\sigma), \quad P_2 = \frac{1}{2}(1+u)\frac{1}{2}(1-i\sigma),$$
$$P_3 = \frac{1}{2}(1-u)\frac{1}{2}(1+i\sigma), \quad P_4 = \frac{1}{2}(1-u)\frac{1}{2}(1-i\sigma). \tag{11}$$

As $iP = -P\sigma$, we have from (9)

$$\partial\psi\sigma + m\psi = 0. \tag{12}$$

Then, because of the fact that e_μ, u, and σ are real, we have $\psi = \psi P_1 \Rightarrow \psi^* = \psi^* P_2 \Rightarrow \mathrm{Re}(\psi)\frac{1}{2}(1+u) = \mathrm{Re}(\psi)$. Thus, the real part of ψ belongs to the minimal left ideal $\mathcal{C}l_{1,3}(\mathbb{R})\frac{1}{2}(1+u)$ in $\mathcal{C}l_{1,3}(\mathbb{R})$, i.e., it is an algebraic spinor in $\mathcal{C}l_{1,3}(\mathbb{R})$ (*real* algebra). Following Lounesto, we define the mother spinor $\Phi = 4\,\mathrm{Re}(\psi)$. Taking the real part of (12),

$$\partial\Phi\sigma + m\Phi = 0, \quad \Phi \in \mathcal{C}l_{1,3}(\mathbb{R})\frac{1}{2}(1+u). \tag{13}$$

This is the Dirac equation written in terms of real algebraic spinors in $\mathcal{C}l_{1,3}(\mathbb{R})$. Following Hestenes's approach, we now obtain an operatorial version of the above equation, but without restricting ourselves to his particular choice of $\{\gamma_\mu\}$ (or, equivalently, of P).

In Subsection 2.1, the passage from algebraic spinors to operator spinors was made by taking the usual even/odd $\mathbb{Z}_2$-grading $\mathcal{C}l_{1,3}(\mathbb{R}) = \mathcal{C}l_{1,3}^+ \oplus \mathcal{C}l_{1,3}^-$ of $\mathcal{C}l_{1,3}(\mathbb{R})$ and then projecting the analogue of (13) in $\mathcal{C}l_{1,3}^+(\mathbb{R})$. This was easily done as the analogue of u (i.e. e_0) was odd and the analogue of σ (i.e. e_{12}) was even. Nevertheless, these conditions no longer hold in the more general setting of this subsection. For instance, in Example 2 below (which is related to the Majorana representation), we have $u = e_{20}$, which is *even* with respect to $\mathcal{C}l_{1,3}^+ \oplus \mathcal{C}l_{1,3}^-$.

So, to consider the spinor as an operatorial object, we will define a convenient $\mathbb{Z}_2$-grading $\mathcal{C}l_{1,3}(\mathbb{R}) = \mathcal{C}l_0 \oplus \mathcal{C}l_1$ such that u is *odd* and σ is *even* with respect to *this* grading. More precisely, we demand that the subspaces $\mathcal{C}l_0$ and $\mathcal{C}l_1$ are such that

$$\mathcal{C}l_0\mathcal{C}l_0 \subseteq \mathcal{C}l_0, \quad \mathcal{C}l_0\mathcal{C}l_1 \subseteq \mathcal{C}l_1, \quad \mathcal{C}l_1\mathcal{C}l_0 \subseteq \mathcal{C}l_1, \quad \mathcal{C}l_1\mathcal{C}l_1 \subseteq \mathcal{C}l_0,$$

and

$$u \in \mathcal{C}l_1, \quad \sigma \in \mathcal{C}l_0.$$ [11]

Note that $\mathcal{C}l_0$ is then a subalgebra of $\mathcal{C}l_{1,3}(\mathbb{R})$.

[11] As $P = \frac{1}{2}(1+u)\frac{1}{2}(1+i\sigma) = \frac{1}{2}(1+u)\frac{1}{2}(1+iu\sigma)$, we can redefine σ as $\sigma' = u\sigma$. As $\sigma' \in \mathcal{C}l_0 \Leftrightarrow \sigma \in \mathcal{C}l_1$, we see that the essential assumption about σ is that it has definite parity (with respect to $\mathcal{C}l_0 \oplus \mathcal{C}l_1$). As a matter of fact, even this assumption can be weakened at the expenses of simplicity.

To each $\mathbb{Z}_2$-grading $\mathcal{C}l_{1,3}(\mathbb{R}) = \mathcal{C}l_0 \oplus \mathcal{C}l_1$ there is a corresponding grading automorphism α such that $\mathcal{C}l_i = \{a \in \mathcal{C}l_{1,3}(\mathbb{R}) : \alpha(a) = (-1)^i a\}$, $i = 0, 1$. We define projections $\pi_i : \mathcal{C}l_{1,3}(\mathbb{R}) \to \mathcal{C}l_i$ by $\pi_i(a) = \frac{a+(-1)^i\alpha(a)}{2}$, $i = 0, 1$. Note that α is related to $\mathcal{C}l_0 \oplus \mathcal{C}l_1$ in exactly the same way as $(\cdot)\hat{}$ is related to $\mathcal{C}l^+_{1,3}(\mathbb{R}) \oplus \mathcal{C}l^-_{1,3}(\mathbb{R})$ (see Introduction for notation).

We refer to $\mathcal{C}l_0$ and $\mathcal{C}l_1$ as the α*-even* and α*-odd* parts of $\mathcal{C}l_{1,3}(\mathbb{R})$. In general, $\mathcal{C}l_0$ and $\mathcal{C}l_1$ are of course different from the usual even ($\mathcal{C}l^+_{1,3}(\mathbb{R})$) and odd ($\mathcal{C}l^-_{1,3}(\mathbb{R})$) parts of $\mathcal{C}l_{1,3}(\mathbb{R})$. Or, in other words, in general we have α different from $(\cdot)\hat{}$.

Now it is possible to obtain an operatorial version of (13) by a standard procedure. Because of the fact that $\Phi = \Phi\frac{1+u}{2}$ and $u \in \mathcal{C}l_1$, it follows that $\Phi = \Phi u$ and $\pi_1(\Phi) = \pi_0(\Phi)u$. Separating (13) in α-even and α-odd parts, we have

$$\pi_0(\partial)\Psi u\sigma + \pi_1(\partial)\Psi\sigma + m\Psi u = 0,$$

where

$$\Psi = \pi_0(\Phi) = \pi_0(4\mathrm{Re}(\psi)). \tag{14}$$

By defining a "projected" Dirac operator $\breve{\partial}$ acting on $\mathcal{C}l_0$ by $\breve{\partial}(\cdot) = \pi_0(\partial)(\cdot)u + \pi_1(\partial)(\cdot)$, the above equation simplifies to

$$\breve{\partial}\Psi\sigma + m\Psi u = 0, \quad \Psi \in \mathcal{C}l_0. \tag{15}$$

This is the generalized operatorial version of the Dirac equation we were looking for.

Example 1 (Standard Representation). Starting with the gamma matrices in the standard representation $\{\gamma^{\mathrm{st}}_\mu\}$, one *calculates* $P = P_{\mathrm{st}} = \frac{1}{2}(1 + e_0)\frac{1}{2}(1 + ie_{12})$, giving $u = e_0$ and $\sigma = e_{12}$. We see that $e_0(e_{12})$ is already odd (even) in the usual $\mathbb{Z}_2$-grading of $\mathcal{C}l_{1,3}(\mathbb{R})$, so we can take $\mathcal{C}l_0 = \mathcal{C}l^+_{1,3}(\mathbb{R})$ and $\mathcal{C}l_1 = \mathcal{C}l^-_{1,3}(\mathbb{R})$. Also, $\breve{\partial} = \partial$. Then, (15) leads to

$$\partial\Psi e_{12} + m\Psi e_0 = 0, \quad \Psi \in \mathcal{C}l^+_{1,3}(\mathbb{R}),$$

which is the Dirac–Hestenes equation, as expected.

Example 2 (Majorana Representation). If we start with the gamma matrices in the Majorana representation (Itzykson and Zuber, 1980):

$$\gamma^{\mathrm{mj}}_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^{\mathrm{mj}}_1 = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}, \quad \gamma^{\mathrm{mj}}_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix},$$
$$\gamma^{\mathrm{mj}}_3 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix},$$

a direct calculation yields $P = P_{\mathrm{mj}} = \frac{1}{2}(1 + e_{20})\frac{1}{2}(1 + ie_1)$, i.e., $u = e_{20}$ and $\sigma = e_1$. The next step is to define a $\mathbb{Z}_2$-grading such that e_{20} is α-odd and e_1 is α-even.

There are plenty of them, but a convenient choice is given by

	$\mathcal{C}l_0$	$\mathcal{C}l_1$
0-vectors	1	
1-vectors	e_1, e_2, e_3	e_0
2-vectors	e_{12}, e_{23}, e_{31}	e_{10}, e_{20}, e_{30}
3-vectors	e_{123}	$e_{012}, e_{023}, e_{031}$
4-vectors		e_{0123}

where the columns give basis for the vector spaces $\mathcal{C}l_0$ and $\mathcal{C}l_1$.[12] Note that $\mathcal{C}l_0 \cong \mathcal{C}l_{0,3}$ and thus

$$\mathcal{C}l_0 \cong \mathbb{H} \oplus \mathbb{H} \quad \text{(as real algebras).}$$

Hence, this algebra is *not isomorphic* to the one in the Hestenes's case, which was given by $\mathcal{C}l^+_{1,3}(\mathbb{R}) \cong \mathcal{C}l_{3,0}(\mathbb{R}) \cong \mathcal{M}(2, \mathbb{C}) \cong$ *Pauli algebra*.

It is important to note that not all vectors e_μ are α-odd in this $\mathbb{Z}_2$-grading. As $\pi_0(\partial) = e_k\partial^k$ and $\pi_1(\partial) = e_0\partial^0$, we have $\breve{\partial}(\cdot) = e_k\partial^k(\cdot)e_{20} + e_0\partial^0(\cdot)$. From (15), the Dirac equation in this case is therefore

$$e_0\partial^0\Psi + e_k\partial^k\Psi e_{20} = m\Psi e_{20}e_1, \quad \Psi \in \mathcal{C}l_0.$$

After right-multiplying by e_0 and noting that $e_0\Psi\, e_0 = \hat{\Psi}$, we finally have

$$\partial^0\hat{\Psi} + e_k\partial^k\Psi e_2 = m\Psi e_{12}, \quad \Psi \in \mathcal{C}l_0 = \mathcal{C}l_{0,3} \cong \mathbb{H} \oplus \mathbb{H}.$$

Example 3 (Chiral Representation). We now consider the chiral representation $\rho_{\text{ch}}(e_\mu) = \gamma^{\text{ch}}_\mu$, where (Itzykson and Zuber, 1980):

$$\rho_{\text{ch}}(e_0) = \gamma^{\text{ch}}_0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \qquad \rho_{\text{ch}}(e_k) = \gamma^{\text{ch}}_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3.$$

A straightforward calculation then gives $P = P_{\text{ch}} = \frac{1}{2}(1 + e_{30})\frac{1}{2}(1 + ie_{12})$, i.e., $u = e_{30}$ and $\sigma = e_{12}$.

Before we derive the operatorial version of the Dirac equation for this case, let us consider the chirality operator in this context. Remember that the chirality operator acting on column spinors is given by Bjorken and Drell (1964) $ch[|\psi\rangle] = \gamma_5|\psi\rangle = -i\gamma_{0123}|\psi\rangle$. By applying ρ^{-1}, we have the corresponding expression for

[12] An analytic expression for the corresponding automorphism grading is, for instance, $\alpha(\eta) = e_{123}\eta e_{123}$.

algebraic spinors: $ch[\psi] = ie_{0123}\psi = e_{0123}\psi\sigma = e_{0123}\psi u\sigma$, for $\psi = \psi u$. Thus $ch[\psi] = e_{0123}\psi e_{30}e_{12} = -e_{0123}\psi e_{0123} = \hat{\psi}$. After applying the isomorphism (14) between algebraic and operator spinors (which is the isomorphism ι discussed in the Appendix), we have $ch[\Psi] = 4\pi_0(\mathrm{Re}(\hat{\psi}))$, which is simply $4\pi_0(\mathrm{Re}(\psi))\hat{} = \hat{\Psi}$ provided that the grading automorphisms $\hat{}$ and α commute. As the e_μ's have definite $\hat{}$-parity and generate $\mathcal{C}l_{1,3}(\mathbb{R})$, this condition holds if and only if each e_μ has definite α-parity (of course, some of the e_μ's can be α-odd and some of them α-even). Thus, this condition is extremely easy to be fulfilled.

A convenient choice for the $\mathbb{Z}_2$-grading here is given by

	$\mathcal{C}l_0$	$\mathcal{C}l_1$
0-vectors	1	
1-vectors	e_0	e_1, e_2, e_3
2-vectors	e_{12}, e_{23}, e_{31}	e_{01}, e_{02}, e_{03}
3-vectors	$e_{012}, e_{023}, e_{031}$	e_{123}
4-vectors		e_{0123}

(16)

Note that $u = e_{30}$ is α-odd, $\sigma = e_{12}$ is α-even, and all of the e_μ's have definite α-parity. As discussed earlier, we then have

$$ch[\Psi] = \hat{\Psi}, \tag{17}$$

giving a particularly simple form for the chirality operator.

We now further decompose $\mathcal{C}l_0$ in its usual ($\wedge$-) even and ($\wedge$-) odd parts $\mathcal{C}l_0^{\pm} := \mathcal{C}l_0 \cap \mathcal{C}l_{1,3}^{\pm}$. It follows from (17) that

$$\mathcal{C}l_0^{+} = [\text{space of right handed spinors}],$$

$$\mathcal{C}l_0^{-} = [\text{space of left handed spinors}],$$

which gives a neat characterization of chirality for operator spinors.

Returning to the Dirac equation and proceeding as in Example 2, we obtain

$$e_0\partial^0\Psi e_{30} + e_k\partial^k\Psi = -m\Psi e_{30}e_{21}, \quad \Psi \in \mathcal{C}l_0.$$

After a little algebra, this gives

$$-\partial^0\Psi e_{12} + (e_{23}\partial^1 + e_{31}\partial^2 + e_{12}\partial^3)\hat{\Psi} = me_0\Psi, \quad \Psi \in \mathcal{C}l_0, \tag{18}$$

where we used the fact that e_0 commutes with $\mathcal{C}l_0$. This is the operatorial version of the Dirac equation for this case. In the following section, we explore the above expression further.

2.3. Quaternionic Representations of the Gamma Matrices

The above form of the Dirac equation gives a natural way to obtain representations of the gamma matrices in terms of the enhanced $\mathbb{H}$-general linear group $GL(2, \mathbb{H}) \cdot \mathbb{H}^*$ (Harvey, 1990), which comprises both a quaternionic matrix multiplication from the left and a quaternionic scalar multiplication from the right.

2.3.1. Chiral-Like Representation

Let us project (18) in $\mathcal{C}l_0^{\pm}$,

$$-\partial^0\Psi_+ e_{12} + (e_{23}\partial^1 + e_{31}\partial^2 + e_{12}\partial^3)\Psi_+ = me_0\Psi_- \quad \text{in}\, \mathcal{C}l_0^+,$$
$$-\partial^0\Psi_- e_{12} - (e_{23}\partial^1 + e_{31}\partial^2 + e_{12}\partial^3)\Psi_- = me_0\Psi_+ \quad \text{in}\, \mathcal{C}l_0^-.$$

After right-multiplying the second equation by e_0 and defining $\chi := \Psi_+$, $\eta := \Psi_- e_0$ (note that $\chi, \eta \in \mathcal{C}l_0^+$), we have

$$-\partial^0\chi e_{12} + (e_{23}\partial^1 + e_{31}\partial^2 + e_{12}\partial^3)\chi = m\eta,$$
$$-\partial^0\eta e_{12} - (e_{23}\partial^1 + e_{31}\partial^2 + e_{12}\partial^3)\eta = m\chi,$$

which are equations within $\mathcal{C}l_0^+ = \text{span}_{\mathbb{R}}\{1, e_{12}, e_{23}, e_{31}\}$. This algebra is isomorphic to $\mathbb{H}$ through $1 \mapsto 1$, $e_{23} \mapsto i$, $e_{31} \mapsto j$, $e_{12} \mapsto k$, where i, j, and k denote the imaginary quaternionic units. To simplify the notation, let $\mathrm{i}_1 = i$, $\mathrm{i}_2 = j$, and $\mathrm{i}_3 = k$ and let χ and η be (by abuse of notation) the quaternions corresponding to χ and η by the above map. Then

$$-\partial^0\chi k + \mathrm{i}_l\partial^l\chi = m\eta,$$
$$-\partial^0\eta k - \mathrm{i}_l\partial^l\eta = m\chi,$$

(note that these equations decouple in the limit $m \to 0$, as expected). In terms of $\binom{\chi}{\eta} \in \mathbb{H}^2$,

$$-\partial^0\begin{pmatrix}\chi\\ \eta\end{pmatrix}k + \partial^l\begin{pmatrix}\mathrm{i}_l & 0\\ 0 & -\mathrm{i}_l\end{pmatrix}\begin{pmatrix}\chi\\ \eta\end{pmatrix} = m\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi\\ \eta\end{pmatrix}.$$

In the Appendix, we discuss in detail the relationship among the different definitions of spinors considered here, and we obtain expressions for the momentum operator and scalar product in terms of operator spinors. From (24) and $\sigma = e_{12}$, it follows that the momentum operator acting on $\binom{\chi}{\eta}$ is given by $\mathbf{p}_\mu[\binom{\chi}{\eta}] = -\partial_\mu\binom{\chi}{\eta}k$. So, the above equation can be written as

$$\mathbf{p}^0\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\chi\\ \eta\end{pmatrix} + \mathbf{p}^l\begin{pmatrix}0 & -\mathrm{i}_l\\ \mathrm{i}_l & 0\end{pmatrix}\begin{pmatrix}\chi\\ \eta\end{pmatrix}k = m\begin{pmatrix}\chi\\ \eta\end{pmatrix}.$$

Let us define $\gamma_\mu : \mathbb{H}^2 \to \mathbb{H}^2$ by

$$\gamma_0 \begin{pmatrix} \chi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix}, \qquad \gamma_l \begin{pmatrix} \chi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}_l \\ \mathrm{i}_l & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} k, \tag{19}$$

where $l = 1, 2, 3$. Note that

(i) γ_1 comprises both a matrix multiplication from the left and a scalar multiplication from the right, i.e., $\gamma_l \in GL(2, \mathbb{H}) \cdot \mathbb{H}^*$, $l = 1, 2, 3$;
(ii) $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$, where the product $\gamma_\mu\gamma_\nu$ is given by composition;
(iii) $[\gamma_\mu, \mathbf{p}_\nu] = 0$, for γ_μ commutes with right multiplication by k; and
(iv) both $\mathbf{p}_\mu$ and γ_0 are Hermitian operators (see below the expression of the scalar product).

After substituting γ_μ in the above equation, we find

$$\gamma_\mu \mathbf{p}^\mu \begin{pmatrix} \chi \\ \eta \end{pmatrix} = m \begin{pmatrix} \chi \\ \eta \end{pmatrix},$$

which is the Dirac equation in $\mathbb{H}^2$ with a $GL(2, \mathbb{H}) \cdot \mathbb{H}^*$-chiral representation (19) for the gamma matrices. This should be compared with de Leo (2001).

Let us denote quaternionic conjugation by a bar, i.e., given $q = q^0 + q^l \mathrm{i}_l$, we have $\overline{q} = q^0 - q^l \mathrm{i}_l$. A straightforward calculation shows that the scalar product (25) (see Appendix) between spinors becomes in this formalism:

$$\langle\psi_1|\psi_2\rangle = \mathrm{proj}_{\mathbb{C}'}(\overline{\chi}_1\chi_2 + \overline{\eta}_1\eta_2), \tag{20}$$

where $\binom{\chi_i}{\eta_i} \leftrightarrow |\psi_i\rangle$, $i = 1, 2$, and $\mathrm{proj}_{\mathbb{C}}'$ projects $\mathbb{H}$ into its complex subspace $\mathbb{C}' = \mathrm{span}_{\mathbb{R}}\{1, -k\} \subseteq \mathbb{H}$. In other words, $\mathbb{C}'$ is a complex subspace of $\mathbb{H}$ with imaginary unit $\sqrt{-1}$ given by $-k$. Note that, if we had chosen to include $|\psi\rangle$ in the fourth column of $\mathcal{M}(4, \mathbb{C})$ (instead of (4)), then we would have $\sqrt{-1} \leftrightarrow k$. More generally, by redefining the above isomorphism between $\mathcal{C}l_0^+$ and $\mathbb{H}$, we could identify subspaces $\mathbb{C}' \subseteq \mathbb{H}$ with imaginary unit $\sqrt{-1}$ corresponding to *any* imaginary unit quaternion.

Observe that (20) is the quaternionic scalar product between $\binom{\chi_1}{\eta_1}$ and $\binom{\chi_2}{\eta_2}$ projected to a complex subspace $\mathbb{C}' \subseteq \mathbb{H}$. Therefore, we have just shown that in this formalism one can *derive* the ad hoc complex projection for the scalar product in Rotelli (1989).

2.3.2. Standard-Like Representation

Another interesting decomposition of $\mathcal{C}l_0$ can be given by noting that, in this case, $\mathcal{C}l_0$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ (as real algebras). To see this, let us pick the mutually annihilating central idempotents $f_\pm = \frac{1}{2}(1 \pm e_0)$ in $\mathcal{C}l_0$. Then we have immediately $\mathcal{C}l_0 = \mathcal{C}l_0 f_+ \oplus \mathcal{C}l_0 f_-$, where $\oplus$ denotes direct sum of algebras. Each factor $\mathcal{C}l_0 f_\pm$ is easily seen to be isomorphic to $\mathbb{H}$ by the map $\mathcal{C}l_0 f_\pm \to \mathbb{H}$, $f_\pm \mapsto 1$, $e_{23} f_\pm \mapsto i$, $e_{31} f_\pm j$, $e_{12} f_\pm \mapsto k$. As the subalgebras $\mathcal{C}l_0 f_\pm$ are orthogonal (i.e. $\mathcal{C}l_0 f_\pm \mathcal{C}l_0 f_\mp = 0$), it is natural to define $\xi : \mathcal{C}l_0 \to \mathbb{H} \oplus \mathbb{H}$ by

$$\xi(f_+) = (1, 0), \quad \xi(e_{23} f_+) = (i, 0), \quad \xi(e_{31} f_+) = (j, 0), \quad \xi(e_{12} f_+) = (k, 0),$$
$$\xi(f_-) = (0, 1), \quad \xi(e_{23} f_-) = (0, i), \quad \xi(e_{31} f_-) = (0, j), \quad \xi(e_{12} f_-) = (0, k).$$

Or, in terms of the elements in (16),

$$\xi(1) = (1, 1), \quad \xi(e_0) = (1, -1),$$
$$\xi(e_{23}) = (i, i), \quad \xi(e_{31}) = (j, j), \quad \xi(e_{12}) = (k, k),$$
$$\xi(e_{023}) = (i, -i), \quad \xi(e_{031}) = (j, -j), \quad \xi(e_{012}) = (k, -k).$$

We note in passing that given $\Psi \in \mathcal{C}l_0$, with $\xi(\Psi) = (q_1, q_2)$, the reversion $\tilde{}$ and the $\wedge$-parity are given in $\mathbb{H} \oplus \mathbb{H}$ by $\xi(\hat{\Psi}) = (q_2, q_1)$ and $\xi(\tilde{\Psi}) = (\bar{q}_1, \bar{q}_2)$.

Let us pause to consider the positive/negative energy states. For the sake of simplicity, let us consider the electron at rest. Then, the energy operator on column spinors is given by $\mathcal{E}[|\psi\rangle] = m\gamma_0|\psi\rangle$. This induces $\mathcal{E}[\psi] = me_0\psi$ on algebraic spinors. By applying the isomorphism ι and (23) (see Appendix), we have $\mathcal{E}[\Psi] = me_0\Psi$ for operator spinors. This gives us $\mathcal{E}[(q_1, q_2)] = m\xi(e_0\Psi) = m(1, -1)(q_1, q_2) = (mq_1, -mq_2)$ at the level of $\mathbb{H} \oplus \mathbb{H}$. Thus, states of the form $(q, 0)$ and $(0, q)$ can be thought of as positive and negative energy spinors respectively. Therefore, it is not a surprise that a standard-like $GL(2, \mathbb{H}) \cdot \mathbb{H}^*$-representation for the gamma matrices can be obtained in this context.

Applying ξ to (18),

$$-\partial^0(q_1, q_2)(k, k) + (\mathrm{i}_l, \mathrm{i}_l)\partial^l(q_2, q_1) = m(1, -1)(q_1, q_2).$$

Or, in terms of $\binom{q_1}{q_2} \in \mathbb{H}^2$,

$$-\partial^0 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} k + \partial^l \begin{pmatrix} 0 & \mathrm{i}_l \\ \mathrm{i}_l & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} = m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

We have again $\mathbf{p}_\mu \left[\binom{q_1}{q_2}\right] = -\partial_\mu \binom{q_1}{q_2} k$. So,

$$\mathbf{p}^0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \mathbf{p}^l \begin{pmatrix} 0 & \mathrm{i}_l \\ -\mathrm{i}_l & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} k = m \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

or

$$\gamma_\mu \mathbf{p}^\mu \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = m \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

where $\gamma_\mu : \mathbb{H}^2 \to \mathbb{H}^2$, $\mu = 0, 1, 2, 3$ are now defined by

$$\gamma_\mathbf{0} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \gamma_\mathbf{l} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{i}_l \\ -\mathrm{i}_l & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} k,$$

where $l = 1, 2, 3$. This gives another $GL(2, \mathbb{H}) \cdot \mathbb{H}^*$-representation of the gamma matrices (now standard-like), again to be compared with de Leo (2001).

2.4. Lagrangian Formulation

We now briefly discuss a Lagrangian formulation for the multivector Dirac equations considered here. In terms of column spinors, the usual Lagrangian for the Dirac theory is given by $\mathcal{L} = \frac{i}{2}(\langle\bar{\psi}|\gamma^\mu\partial_\mu|\psi\rangle - \partial_\mu\langle\bar{\psi}|\gamma^\mu|\psi\rangle) - m\langle\bar{\psi} \mid \psi\rangle$. By employing the spinor maps from the Appendix, we obtain the corresponding expression for operator spinors:

$$\mathcal{L} = -\langle\tilde{\Psi}\breve{\partial}(\Psi)\sigma\xi_0\rangle_0 - m\langle\tilde{\Psi}\Psi u\xi_0\rangle_0, \quad \Psi \in \mathcal{C}l_0,$$

where we define $\xi_0 = ue_0$ if $e_0 \in \mathcal{C}l_0$ and $\xi_0 = e_0$ if $e_0 \in \mathcal{C}l_1$ (note that ξ_0 is always α-odd). The equations of motion derived from the above Lagrangian are indeed the multivector Dirac equation (15), as expected.

It is important to note that this derivation owes much of its simplicity to the fact that we had already identified the spinor space $\mathcal{C}l_0$ (which in turn determines the modified Dirac operator $\breve{\partial}$) in Section 2.2.

3. BACK TO HESTENES'S FORM

In the previous section, we have obtained multivector Dirac equations corresponding to different representations of the gamma matrices. These equations and the spinor spaces associated to them are, in general, distinct from those obtained in the context of the Dirac–Hestenes equation. Now we show that, by allowing arbitrary $\mathbb{Z}$-gradings for the *vector space* structure of $\mathcal{C}l_{1,3}(\mathbb{R})$, we can always obtain a multivector equation similar to the Dirac–Hestenes one, and with spinor space isomorphic to the Pauli algebra. By an arbitrary $\mathbb{Z}$-grading we mean one that is not necessarily in the form $\mathcal{C}l_{1,3}(\mathbb{R}) = \oplus_{k=0}^{4}\Lambda_k(\mathbb{M})$, where $\mathbb{M} \cong \mathbb{R}^{1,3}$ is the tangent space of space-time at a given point.

Let $\{e_\mu\}$ be an orthonormal basis of $\mathbb{M}$ corresponding to some observer's reference frame. In Section 2.2, we showed that to representations $\{\gamma_\mu\}$ of the gamma matrices there correspond different operatorial versions of the Dirac equation. To do that, we first defined the γ_μ-dependent representation $\rho : \mathcal{C}l_{1,3}(\mathbb{C}) \to \mathcal{M}(4, \mathbb{C})$, $\rho(e_\mu) = \gamma_\mu$, and then transferred the Dirac equation to the ideal $\mathcal{C}l_{1,3}(\mathbb{C})P$, where $P = \rho^{-1}(\mathsf{P})$ and P is given by (4). After that we showed that by taking a convenient $\mathbb{Z}_2$-grading $\mathcal{C}l_{1,3}(\mathbb{R}) = \mathcal{C}l_0 \oplus \mathcal{C}l_1$ we could obtain a corresponding operatorial version of the Dirac equation in the algebra $\mathcal{C}l_0$. As we have already seen, P can be written as

$$P = \frac{1}{2}(1 + u)\frac{1}{2}(1 + i\sigma),$$

where $u = S^{-1}e_0S$ and $\sigma = S^{-1}e_{12}S$.

Let $\{E_\mu\}$ be defined by $E_\mu = S^{-1}e_\mu S$. Of course, $E_\mu E_\nu + E_\nu E_\mu = 2g_{\mu\nu}$, i.e., $\{E_\mu\}$ is still an ONS. However, in general, the elements E_μ *are not vectors* in M.[13] As a matter of fact, E_μ can even have a nonzero imaginary part for certain choices of S. Let us define the reversion $(\cdot)^R$ and parity $(\cdot)^P$ operators *relative to* the ONS $\{E_\mu\}$ by $(E_\mu)^R = E_\mu$, extended to $\mathcal{C}l_{1,3}(\mathbb{C})$ as an anti-automorphism, and $(E_\mu)^P = -E_\mu$, extended to $\mathcal{C}l_{1,3}(\mathbb{C})$ as an automorphism. Of course, $(\cdot)^R$ and $(\cdot)^P$ define a $\mathbb{Z}$-grading for the underlying vector space structure of $\mathcal{C}l_{1,3}(\mathbb{R})$ by

$$\mathcal{C}l_{1,3}(\mathbb{C}) = \bigoplus_{k=0}^{4} C_k,$$

where $C_0 = \mathbb{C}$ (scalars), $C_1 = \{a : a^P = -a \text{ and } a^R = a\}$, $C_2 = \{a : a^P = a \text{ and } a^R = -a\}$, $C_3 = \{a : a^P = -a \text{ and } a^R = -a\}$, and $C_4 = \text{span}_{\mathbb{C}}\{E_{0123}\}$. Note that C_4 is the orthogonal space to C_0 inside $\{a : a^P = a \text{ and } a^R = a\}$.

Let $\rho : \mathcal{C}l_{1,3}(\mathbb{C}) \to \mathcal{M}(4, \mathbb{C})$ be the isomorphism defined by $\rho(E_\mu) = \gamma_\mu^{\text{st}}$. Proceeding as in Section 2.2 but now with $P = \frac{1}{2}(1 + E_0)\frac{1}{2}(1 + iE_{12})$, i.e., with $u = E_0$ and $\sigma = E_{12}$, we obtain the analogue of (13),

$$\mathfrak{D}\Phi E_{12} + m\Phi = 0,$$

where $\Phi := 4\,\text{Re}(\psi)$ and $\mathfrak{D} := E_\mu \partial^\mu$ (note that $\mathfrak{D}$ is not the usual Dirac operator ∂, for E_μ is not a vector in $\mathbb{M}$, $\mu = 0, 1, 2, 3$). The above $\mathbb{Z}$-grading for the underlying *vector space structure* of $\mathcal{C}l_{1,3}(\mathbb{R})$ induces a $\mathbb{Z}_2$-grading for the *algebra structure* of $\mathcal{C}l_{1,3}(\mathbb{R})$ by

$$\mathcal{C}l_{1,3}(\mathbb{R}) = \mathcal{C}l_0 \oplus \mathcal{C}l_1,$$

where $\mathcal{C}l_0 := \oplus_{k\,\text{even}} C_k$ and $\mathcal{C}l_1 := \oplus_{k\,\text{odd}} C_k$. With respect to the notation in Section 2.2, we can take now $\alpha = (\cdot)^P$ and $\breve{\partial} = \mathfrak{D}$, resulting in the following operatorial form of the Dirac equation:

$$\mathfrak{D}\Psi E_{21} = m\Psi E_0, \tag{21}$$

with $\Psi \in \mathcal{C}l_0 = C_0 \oplus C_2 \oplus C_4$ and $\mathfrak{D} := E_\mu \partial^\mu$. We see that (21) has the form of the Dirac–Hestenes equation but with the difference that every quantity here is conjugated to the corresponding Hestenes quantity by a similarity transformation. For instance, the algebra of operator spinors is $\mathcal{C}l_0 = S^{-1}\mathcal{C}l_{1,3}^+ S$, and thus it is always isomorphic to $\mathcal{C}l_{1,3}^+ \cong \mathcal{M}(2, \mathbb{C}) \cong$ [Pauli algebra]. This should be compared with our former examples.

[13] Note that the scalar product between elements of this ONS is given by $g(E_\mu, E_\nu) = \langle \tilde{E}_\mu E_\nu \rangle_0 = \langle \tilde{S} e_\mu \tilde{S}^{-1} S^{-1} e_\nu S \rangle_0 = \langle e_\mu (S\tilde{S})^{-1} e_\nu S\tilde{S} \rangle_0$. When $S \in Spin_+(1, 3)$ the map $e_\mu \to S^{-1} e_\mu S$ is an isometry in $\mathbb{R}^{1,3}$ and $S\tilde{S} = 1$ (Lounesto, 1996). Therefore, in this case we have $g(E_\mu, E_\nu) = g(e_\mu, e_\nu) = g_{\mu\nu}$. However, in the more general setting of this section, this is false: we do have $E_\mu E_\nu + E_\nu E_\mu = 2g_{\mu\nu}$, but in general $E_\mu \notin \mathbb{R}^{1,3}$ and $g(E_\mu, E_\nu) \neq g_{\mu\nu}$.

As a consequence, the spinor $\Psi \in \mathcal{C}l_0$ still has a polar decomposition $\Psi = \sqrt{\varrho}e^{\frac{\beta}{2}E_{0123}}R = S^{-1}(\sqrt{\varrho}e^{e_{0123}}R')S$, where R' (and not R) covers a restricted Lorentz transformation of the space-time. Also, we should not interpret the quantity $\tilde{\Psi}E_0\Psi$ as the current density as in the Hestenes approach. Now this current is given by $j = j^\mu e_\mu = S(j^\mu E_\mu)S^{-1}$.

Note that the ONS $\{E_\mu\}$ does span a four-dimensional real vector space $W := \text{span}_{\mathbb{R}}\{E_0, E_1, E_2, E_3\}$, on which we can define the metric $h(x, y) = \frac{1}{2}(xy + yx)$, $x, y \in W$ (in this expression, the products xy and yx are just the Clifford products inherited from $\mathcal{C}l_{1,3}(\mathbb{C})$). As a result, W is isometric to $\mathbb{M}$ and therefore $\mathcal{C}l_{\mathbb{C}}(W) \cong \mathcal{C}l_{1,3}(\mathbb{C})$. However, the real subspace W is a combination of multivectors of *different grades* of $\mathbb{M}$. Observe also that the polar decomposition for Ψ above gives $\Psi = \sqrt{\varrho}e^{\frac{\beta}{2}E_{0123}}R$ with $R \in \mathcal{S}pin_+(W) = \{a : a^P = a \text{ and } a^R a = 1\}$. Of course, $\mathcal{S}pin_+(W)$ is the double cover of the restricted isometry group $SO_+(W)$ of W and $SO_+(W) \cong \mathcal{L}_+^\uparrow$ (as groups). Nevertheless, $\mathcal{L}_+^\uparrow$ and $SO_+(W)$ have very different geometrical interpretations. In particular, the symmetry group of space-time is $\mathcal{L}_+^\uparrow$, and not $SO_+(W)$.

Let us now consider only representations of the gamma matrices which give real E_μ,[14] $\mu = 0, 1, 2, 3$. In this case, W is a vector subspace of $\mathcal{C}l_{1,3}(\mathbb{R})$ isometric to $\mathbb{M}$ and $\mathcal{C}l_{\mathbb{R}}(W) \cong \mathcal{C}l_{1,3}(\mathbb{R})$ (note that the standard representation leads to the strict equality $W = \mathbb{M}$). Therefore, as we have advanced in the Introduction, different representations $\{\gamma_\mu\}$ determine different slices W of the space-time algebra $\mathcal{C}l_{1,3}(\mathbb{R})$, each of them corresponding to a copy of $\mathbb{M}$. Thus, we are naturally led to speculate about a possible connection between this work and Pezzaglia's polydimensional physics program (Pezzaglia, 1999).

APPENDIX: SPINOR MAPS

In this section, we elaborate on the correspondence between algebraic and operator spinors. Let $\iota : \mathcal{C}l_{1,3}(\mathbb{C})P \to \mathcal{C}l_0$ be the map (14) relating algebraic spinors to operator, spinors i.e., $\iota(\psi) = \Psi = 4\pi_0(\text{Re}(\psi))$. Although $\mathcal{C}l_0$ is a real algebra, it has a natural complex structure[15] $J : \Psi \mapsto -\Psi\sigma$. It follows that, with this complex structure, ι is a *complex isomorphism*. To prove that, let us first exhibit an inverse for ι.

As $\psi = \psi P_1$ (cf (11)), we have $2\,\text{Re}(\psi) = (\psi + \psi^*) = \psi P_1 + \psi^* P_2$. As $u \in \mathcal{C}l_1$ and $\sigma \in \mathcal{C}l_0$, we have $\alpha(P_1) = P_3$ and $\alpha(P_2) = P_4$. Thus $\Psi = 4\pi_0(\text{Re}(\psi)) = \psi P_1 + \psi^* P_2 + \alpha(\psi)P_3 + \alpha(\psi^*)P_4$. After right-multiplying by P_1, we have (remember that $\psi = \psi P_1$)

$$\psi = \Psi P_1, \tag{22}$$

[14] This is equivalent to our assumption that u and σ are real in (10).

[15] A complex structure on a vector space V is an endomorphism J on V such that $J^2 = -1_V$.

and hence $\iota^{-1}(\Psi) = \Psi P_1$. Obviously, ι^{-1} preserves sums. Also, $\iota^{-1}(J(\Psi)) = \iota^{-1}(-\Psi\sigma) = -\Psi\sigma P_1 = i\Psi P_1 = i\iota^{-1}(\Psi)$ and thus ι^{-1} and ι are complex isomorphisms, as claimed.

Let us now consider the operatorial version for the action of a matrix A on $|\psi\rangle$. Applying the isomorphism ρ of the last section and defining $A = \rho^{-1}(\mathsf{A})$, we have $\mathsf{A}|\psi\rangle \mapsto A\psi$ at the algebraic spinorial level. After decomposing A in its real and complex parts $A = A_R + iA_I$, we have $A\psi = A_R\psi - A_I\psi\sigma$. To go to the operatorial level, we apply the above ι to obtain

$$\iota(A_R\psi) = \begin{cases} A_R\Psi, & \text{if } A_R \in \mathcal{C}l_0, \\ A_R\Psi u, & \text{if } A_R \in \mathcal{C}l_1, \end{cases} \tag{23}$$

with analogous expressions for A_I. Indeed, if $A_R \in \mathcal{C}l_0$, then $\pi_0(\mathrm{Re}(A_R\psi)) = \pi_0(A_R\,\mathrm{Re}(\psi)) = A_R\pi_0(\mathrm{Re}(\psi))$. Similarly, if $A_R \in \mathcal{C}l_1$, then $\pi_0(A_R\,\mathrm{Re}(\psi)) = \pi_0(A_R\,\mathrm{Re}(\psi u)) = \pi_0(A_R\,\mathrm{Re}(\psi)u) = A_R\pi_0(\mathrm{Re}(\psi))u$, where we used the fact that $\psi = \psi u$.

In a more abstract level, we can use (23) to consider $\mathcal{C}l_0$ as the representation space of $\mathcal{C}l_{1,3}(\mathbb{R})$. More precisely, one can define a representation $\mathfrak{ev} : \mathcal{C}l_{1,3}(\mathbb{R}) \to Aut(\mathcal{C}l_0)$ as follows. Given $\varphi \in \mathcal{C}l_{1,3}(\mathbb{R})$ and $\Psi \in \mathcal{C}l_0$, we decompose $\varphi = \varphi_0 + \varphi_1$ with $\varphi_i \in \mathcal{C}l_i$ and we define (Dimakis, 1989):

$$\mathfrak{ev}(\varphi)(\Psi) = \varphi_0\Psi + \varphi_1\Psi u.$$

We note that the "projected" Dirac operator $\breve{\partial}$ defined in the Section 2.2 can be written simply as

$$\breve{\partial} = \mathfrak{ev}(\partial).$$

Let us now express the momentum operator in the operatorial formalism. For column spinors, we have $p_\mu[|\psi\rangle] = i\partial_\mu|\psi\rangle$. Through ρ, we have for the algebraic spinor $\psi : p_\mu[\psi] = i\partial_\mu\psi = -\partial_\mu\psi\sigma$. Then, by applying ι and remembering that σ is α-even:

$$p_\mu[\Psi] = -\partial_\mu\Psi\sigma. \tag{24}$$

Finally, we consider the scalar product of spinors. If $|\theta\rangle$ and $|\psi\rangle$ are column spinors, the usual spinorial part of the scalar product is defined by $\langle\theta \mid \psi\rangle = |\theta\rangle^\dagger|\psi\rangle$. Let Θ and Ψ be the operator spinors corresponding to $\theta = \rho^{-1}(|\theta\rangle)$ and $\psi = \rho^{-1}(|\psi\rangle)$. Then

$$\langle\theta|\psi\rangle = 4\langle\Theta^\dagger\Psi P\rangle_0, \tag{25}$$

where the dagger operation $(\cdot)^\dagger$ on $\mathcal{C}l_{1,3}$ was defined in the Introduction.[16] To prove (25), we first note that the trace in $\mathcal{M}(4,\mathbb{C})$ is related to $\langle\,\rangle_0$ in $\mathcal{C}l_{1,3}(\mathbb{C})$ by $tr(\rho(\psi)) = 4\langle\psi\rangle_0$, for expanding $\psi = a + a^\mu e_\mu + \cdots$, we have

[16] It is easily seen that $A^\dagger = e_0\tilde{A}^*e_0$ for $A \in \mathcal{C}l_{1,3}(\mathbb{C})$, but this fact was not used in this paper.

$\rho(\psi) = aI + a^\mu \gamma_\mu + \cdots \Rightarrow tr(\rho(\psi)) = 4a = 4\langle\psi\rangle_0$. Then $\langle\theta|\psi\rangle = tr(\rho(\theta)^\dagger \rho(\psi)) = 4\langle\theta^\dagger\psi\rangle_0 = 4\langle P^\dagger\Theta^\dagger\Psi P\rangle_0 = 4\langle\Theta^\dagger\Psi P\rangle_0$, where we substituted $\theta = \Theta P$, $\psi = \Psi P$ (from (22)) and we used the fact that $P^\dagger = P$, which is easily seen by the form of $\rho(P) = \mathsf{P}$ in (4).

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